

CONVERGENCE OF THE SPLITTING METHOD FOR SHALLOW WATER EQUATIONS ¹

MARIA MORANDI CECCHI and LUCA SALASNICH

Dipartimento di Matematica Pura ed Applicata
Università di Padova, Via Belzoni 7, I-35131 Padova, Italy
E-mail: mcecchi@math.unipd.it, salasnich@math.unipd.it

Abstract

In this paper we analyze the convergence of the splitting method for shallow water equations. In particular we give an analytical estimation of the time step which is necessary for the convergence and then we study the behaviour of the motion of the shallow water in the Venice lagoon by using the splitting method with a finite element space discretization. The numerical calculations show that the splitting method is convergent if the time step of the first part is sufficiently small and that it gives a good agreement with the experimental data.

1 Introduction

The equations of a Newtonian (viscous) fluid are called *Navier–Stokes equations* and in the case of a inviscid fluid they give the Euler equations. The Navier–Stokes equations are given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \mathbf{f} ,$$

$$\nabla \cdot \mathbf{u} = 0 , \tag{1}$$

¹Invited paper to the VII International Conference on *Artificial Intelligence and Information–Control Systems of Robots*, Institute of Computer Systems, Slovak Academy of Sciences, 10–14 September, Bratislava (1997).

where $p = P/\rho$ is the reduced pressure and $\nu = \mu/\rho$ is the reduced or kinematic viscosity¹.

The Navier–Stokes equations are to be integrated in the space–time domain $\Omega \times]0, T[\subset \mathbf{R}^3 \times \mathbf{R}^+$ once an appropriate set of initial and boundary conditions has been defined. While the former conditions, in general, only need to reproduce a feasible shape of the current solution, the latter are to be set and treated with much care due to the effect they can have on the evolution process^{2,3}.

In the shallow water hypothesis one assumes that the characteristic horizontal scale L for the motion (the wave length) is longer than the average height \bar{h} of the fluid, i.e. $L \gg \bar{h}$. In such hypothesis vertical acceleration and velocity are negligible and the flux becomes almost horizontal⁴.

Definition 1.1 *The total height of the water is given by*

$$h(x_1, x_2, t) = H(x_1, x_2) + \eta(x_1, x_2, t) ,$$

where H is the depth of the water in the stationary condition above the reference level $x_3 = 0$, and η is the time–dependent difference, i.e. the height of the free surface.

The external forces acting on the water are of extreme importance to determine the equations of motion of the system.

Definition 1.2 *The force of gravity is defined as*

$$\mathbf{f}_g = \int_{\Omega} \rho \mathbf{g} \, d\mathbf{x} ,$$

where $\mathbf{g} = [0, 0, -g]^T$ is the vector acceleration of gravity.

If we consider large regions of water it is necessary to include other forces, like the Coriolis force and the Chezy force, which models the friction of the water at the bottom.

Definition 1.3 *The Coriolis force is defined as*

$$\mathbf{f}_{cor} = \int_{\Omega} \omega \wedge \mathbf{u} \, d\mathbf{x} ,$$

where $\omega = [0, 0, \omega_3]^T$ is the rotation vector of the earth and $\omega_3 = k_0$ is called Coriolis coefficient.

Definition 1.4 *The Chezy force is defined as*

$$\mathbf{f}_{ch} = \int_{\Omega} \frac{g\mathbf{u}|u|}{k_1^2 h} d\mathbf{x} ,$$

where $|u| = \sqrt{u_1^2 + u_2^2}$, g is the scalar acceleration of gravity, h is the total height of the water and k_1 is the Chezy coefficient.

The first step to obtain the shallow water equations is to put u_3 and du_3/dt equal to zero in the Navier Stokes equations. Then the third equation can be integrated between $-H$ and η . The system is yet 3-dimensional because the velocities u_1 and u_2 are functions also of the x_3 variable. It is possible to obtain a 2-dimensional system by performing the substitution

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &\rightarrow a(\psi)u_1(x_1, x_2, t) , \\ u_2(x_1, x_2, x_3, t) &\rightarrow a(\psi)u_2(x_1, x_2, t) , \end{aligned}$$

where

$$\psi = \frac{x_3 + H}{\eta + H} \quad \text{with} \quad \int_0^1 a(\psi) d\psi = 1 ,$$

and then by integrating the new equations over the x_3 variable. In this way one can prove the following theorem (see Ref. 5 and 6 for details).

Theorem 1.5 *Let us consider the inviscid shallow water with external forces given by the gravity force and the Coriolis force. The 2-dimensional viscid shallow water equations are*

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_1}{\partial x_1} + \frac{\partial \mathbf{F}_2}{\partial x_2} = \mathbf{R}_s \quad (2)$$

where $\mathbf{U} = [\eta, u_1, u_2]^T$ is the vector of the conservative variables, the flow vectors \mathbf{F}_1 and \mathbf{F}_2 are

$$\mathbf{F}_1 = [Hu_1, g\eta, 0]^T , \quad \mathbf{F}_2 = [Hu_2, 0, g\eta]^T , \quad (3)$$

and the source vector \mathbf{R}_s is given by

$$\mathbf{R}_s = [0, k_0 u_2 - \frac{gu_1|u|}{k_1^2 H}, -k_0 u_1 - \frac{gu_2|u|}{k_1^2 H}]^T , \quad (4)$$

where g is the acceleration of gravity, k_0 and k_1 are the coefficients of Coriolis and Chezy and $H(x_1, x_2)$ is the depth of the water in the stationary condition.

The shallow water equations are to be integrated in the space–time domain $\Omega \times]0, T[\subset \mathbf{R}^2 \times \mathbf{R}^+$ once an appropriate set of initial and boundary conditions has been defined

$$\mathbf{U}(x_1, x_2, 0) = \mathbf{U}_0(x_1, x_2) \quad \forall (x_1, x_2) \in \Omega, \quad (5)$$

$$\mathbf{U}(x_1, x_2, t) = \mathbf{U}_{\partial\Omega}(x_1, x_2, t) \quad \forall (x_1, x_2) \in \partial\Omega, \forall t \in]0, T[. \quad (6)$$

While the initial conditions, in general, only need to reproduce a feasible shape of the current solution, the boundary conditions are to be set and treated with much care due to the effect they can have on the evolution process.

2 The splitting method

A semi–implicit method is adopted for the solution of the system (2), based on the following splitting (see Ref. 7 and 8)

$$\mathbf{F}_i = \mathbf{F}_i^* + \mathbf{F}_i^{**}, \quad i = 1, 2 \quad (7)$$

and

$$\mathbf{U}^{(n+1)} = \mathbf{U}^{(n)} + \Delta\mathbf{U}^* + \Delta\mathbf{U}^{**}, \quad (8)$$

where $\Delta\mathbf{U}^*$ and $\Delta\mathbf{U}^{**}$ are the increments of the solution vector.

In the iterative scheme we put $\mathbf{F}_i^* = 0$ so that $F_i = F_i^{**}$, $i = 1, 2$, and the system (2) can be divided in:

$$\frac{\partial \Delta\mathbf{U}^*}{\partial t} = \mathbf{R}_s. \quad (9)$$

and

$$\frac{\partial \Delta\mathbf{U}^{**}}{\partial t} + \frac{\partial \mathbf{F}_1^{**}}{\partial x_1} + \frac{\partial \mathbf{F}_2^{**}}{\partial x_2} = \mathbf{0}. \quad (10)$$

These equations are integrated in time, in turn, by using an explicit Taylor–Galerkin method⁸ for (9) and an implicit θ –method⁹ for (10).

The equation (9) is discretized in time by using a Taylor expansion to the second order

$$(\Delta\mathbf{U}^*)^{(n+1)} = \tau \left(\frac{\partial \Delta\mathbf{U}^*}{\partial t} \right)^{(n)} + \frac{\tau^2}{2} \left(\frac{\partial^2 \Delta\mathbf{U}^*}{\partial t^2} \right)^{(n)}, \quad (11)$$

where τ is the time step. From

$$\frac{\partial \Delta \mathbf{U}^*}{\partial t} = \mathbf{R}_s \quad \text{and} \quad \frac{\partial^2 \Delta \mathbf{U}^*}{\partial t^2} = \frac{\partial \mathbf{R}_s}{\partial t} = \frac{\partial \mathbf{R}_s}{\partial \Delta \mathbf{U}^*} \frac{\partial \Delta \mathbf{U}^*}{\partial t}, \quad (12)$$

the equation (11) can be written

$$(\Delta \mathbf{U}^*)^{(n+1)} = \tau (\mathbf{R}_s)^{(n)} + \frac{\tau^2}{2} (\mathbf{G} \mathbf{R}_s)^{(n)}, \quad (13)$$

where

$$\mathbf{G} = \frac{\partial \mathbf{R}_s}{\partial \Delta \mathbf{U}^*}. \quad (14)$$

Because of the computational complexity in the evaluation of the right term of equation (13), we use a two-step version of the Taylor–Galerkin algorithm. This is given by an approximation of $(\mathbf{U})^{(n+1/2)}$ and $(\mathbf{R}_s)^{(n+1/2)}$ the Taylor expansion at step $(n + 1/2)$

$$\mathbf{U}^{(n+1/2)} = \mathbf{U}^{(n)} + \frac{\tau}{2} (\mathbf{R}_s)^{(n)}, \quad (15)$$

$$(\mathbf{R}_s)^{(n+1/2)} = (\mathbf{R}_s)^{(n)} + \frac{\tau}{2} \left(\frac{\partial \mathbf{R}_s}{\partial t} \right)^{(n)} = (\mathbf{R}_s)^{(n)} + \frac{\tau}{2} (\mathbf{G} \mathbf{R}_s)^{(n)}, \quad (16)$$

from which we obtain $(\mathbf{G} \mathbf{R}_s)^{(n)}$ as

$$(\mathbf{G} \mathbf{R}_s)^{(n)} = \frac{2}{\tau} [(\mathbf{R}_s)^{(n+1/2)} - (\mathbf{R}_s)^{(n)}]. \quad (17)$$

It follows that $(\Delta \mathbf{U}^*)^{(n+1)}$ can be written as

$$(\Delta \mathbf{U}^*)^{(n+1)} = \tau (\mathbf{R}_s)^{(n+1/2)}, \quad (18)$$

and the explicit scheme of the splitting method results

$$\mathbf{U}^{(n+1)} = \mathbf{U}^{(n)} + \mathbf{R}_s^{(n+1/2)}. \quad (19)$$

The equation (10) is given, in explicit form, by

$$\begin{aligned} \frac{\partial}{\partial t} (\Delta \eta^{**}) + \frac{\partial}{\partial x_1} (H u_1) + \frac{\partial}{\partial x_2} (H u_2) &= 0 \\ \frac{\partial}{\partial t} (\Delta u_1^{**}) + \frac{\partial}{\partial x_1} (g \eta) &= 0 \\ \frac{\partial}{\partial t} (\Delta u_2^{**}) + \frac{\partial}{\partial x_2} (g \eta) &= 0. \end{aligned} \quad (20)$$

The time discretization is obtained by using the θ -method

$$\begin{aligned} (\Delta\eta^{**})^{(n+1)} + \tilde{\tau} \left[\frac{\partial}{\partial x_1} (Hu_1^{(n+\theta_1)}) + \frac{\partial}{\partial x_2} (Hu_2^{(n+\theta_1)}) \right] &= 0 \\ (\Delta u_1^{**})^{(n+1)} + \tilde{\tau} g \frac{\partial}{\partial x_1} \eta^{(n+\theta_2)} &= 0 \\ (\Delta u_2^{**})^{(n+1)} + \tilde{\tau} g \frac{\partial}{\partial x_2} \eta^{(n+\theta_2)} &= 0 \end{aligned} \quad (21)$$

where $\tilde{\tau}$ is the time step, θ_1 and θ_2 are real parameters between 0 and 1 and

$$\begin{aligned} u_i^{(n+\theta_1)} &= u_i^{(n)} + \theta_1 \left[(\Delta u_i^*)^{(n+1)} + (\Delta u_i^{**})^{(n+1)} \right] \\ \eta^{(n+\theta_2)} &= \eta^{(n)} + \theta_2 (\Delta\eta^{**})^{(n+1)} . \end{aligned} \quad i = 1, 2 \quad (22)$$

Note that the term $(\Delta\eta^*)^{(n+1)}$ does not appear because the splitting method does not include variations in the water elevation due to (13). Substituting (22) into (21), we can write

$$\begin{aligned} (\Delta\eta^{**})^{(n+1)} + \tilde{\tau} \theta_1 \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left[H (\Delta u_1^{**})^{(n+1)} \right] &= \\ - \tilde{\tau} \sum_{i=1}^2 \frac{\partial}{\partial x_i} (Hu_1^{(n)}) + \tilde{\tau} \theta_1 \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left[H (\Delta u_1^*)^{(n+1)} \right] & \\ (\Delta u_1^{**})^{(n+1)} + \tilde{\tau} \theta_2 \frac{\partial}{\partial x_1} g \left[(\Delta\eta^{**})^{(n+1)} \right] &= -\tilde{\tau} g \frac{\partial}{\partial x_1} \eta^{(n)} \\ (\Delta u_2^{**})^{(n+1)} + \tilde{\tau} \theta_2 g \frac{\partial}{\partial x_2} \left[(\Delta\eta^{**})^{(n+1)} \right] &= -\tilde{\tau} g \frac{\partial}{\partial x_2} \eta^{(n)} . \end{aligned} \quad (23)$$

Here $(\Delta u_1^{**})^{(n+1)}$ and $(\Delta u_2^{**})^{(n+1)}$ can be obtained from the second and third equation as a function of $(\Delta\eta^{**})^{(n+1)}$ and then substituted in the first equation which becomes

$$\begin{aligned} (\Delta\eta^{**})^{(n+1)} - \tilde{\tau}^2 \theta_1 \theta_2 g \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left[H \frac{\partial}{\partial x_i} (\Delta\eta^{**})^{(n+1)} \right] &= \\ - \tilde{\tau} \left\{ \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left[H \left(u_1^{(n)} + \theta_1 (\Delta u_i^*)^{(n+1)} \right) \right] - \tilde{\tau} \theta_1 g \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(H \frac{\partial}{\partial x_i} \eta^{(n)} \right) \right\} . \end{aligned} \quad (24)$$

3 Time Convergence of the splitting method

In this section we study analytically the time convergence of the the splitting method. Our study is motivated by the stability analysis of multilevel methods for the numerical simulation of turbulent flows performed by Roger Temam. He analyzed a simple model, namely a pair of ordinary coupled differential equations, by using numerical schemes based on different treatments and different time steps for the two variables of the system¹⁰.

In our splitting method, starting from the time discretization of the previous section, we have

$$\begin{aligned}\eta^{(n+1)} &= \eta^{(n)} + (\Delta\eta^*)^{(n+1)} + (\Delta\eta^{**})^{(n+1)} \\ u_1^{(n+1)} &= u_1^{(n)} + (\Delta u_1^*)^{(n+1)} + (\Delta u_1^{**})^{(n+1)} \\ u_2^{(n+1)} &= u_2^{(n)} + (\Delta u_2^*)^{(n+1)} + (\Delta u_2^{**})^{(n+1)}.\end{aligned}\tag{25}$$

We observe that the first component of $\mathbf{R}_s^{(n+1/2)}$ is zero so that $(\Delta\eta^*)^{(n+1)} = 0$.

It follows that the first part of the splitting scheme reads

$$\begin{aligned}\eta^{(n+1)} &= \eta^{(n)} \\ u_1^{(n+1)} &= u_1^{(n)} + (\Delta u_1^*)^{(n+1)} \\ u_2^{(n+1)} &= u_2^{(n)} + (\Delta u_2^*)^{(n+1)}\end{aligned}\tag{26}$$

where

$$(\Delta u_1^*)^{(n+1)} = \alpha u_1^{(n)} + \beta u_2^{(n)}\tag{27}$$

$$(\Delta u_2^*)^{(n+1)} = -\beta u_1^{(n)} + \alpha u_2^{(n)}\tag{28}$$

with $\alpha = (-\frac{\tau^2 k_0^2}{2} - \tau D + \frac{\tau^2 D^2}{2})$, $\beta = (\tau k_0 - \tau^2 D)$ and $D = g|u|/(k_1^2 H)$. Here we suppose that D is constant during the iterative process. The iteration scheme can be written as

$$\begin{pmatrix} \eta^{(n+1)} \\ u_1^{(n+1)} \\ u_2^{(n+1)} \end{pmatrix} = J^* \begin{pmatrix} \eta^{(n)} \\ u_1^{(n)} \\ u_2^{(n)} \end{pmatrix}\tag{29}$$

where

$$J^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \alpha & \beta \\ 0 & -\beta & 1 + \alpha \end{pmatrix}.\tag{30}$$

Remark 3.1 *We observe that the first part of the splitting scheme and its matrix J^* do not depend on θ_1 and θ_2 .*

Let us study the properties of the first part of the splitting scheme generated by the matrix J^* .

Definition 3.2 *The spectral radius of the matrix J^* is given by*

$$\rho(J^*) = \max\{|\lambda_i|, i = 1, 2, 3\}$$

where λ_i , $i = 1, 2, 3$, are the eigenvalues of the matrix J^* .

By solving the eigenvalue equation we find that the eigenvalues of J^* are $\lambda_1 = 0$ and $\lambda_{2,3} = (1 + \alpha) \pm i\beta$.

Definition 3.3 *The iteration scheme induced by the matrix J^* is called convergent if the spectral radius is such that $\rho(J^*) < 1$.*

Now we can prove the following theorem.

Theorem 3.4 *The first part of the splitting method, induced by the matrix J^* , is not convergent for $D = 0$. For $D \neq 0$ it is convergent if and only if the following inequality is satisfied*

$$(k_0^4 + D^2(4 - k_0^2) + 4D^2)\tau^3 - 4D(D^2 + 2k_0 - k_0^2)\tau^2 + 8D^2\tau - 8D < 0, \quad (31)$$

where $D = g|u|/(k_1^2 H)$.

Proof. The convergence condition for the iterative process is such that the spectral radius of J^* is less than one. We have seen that the eigenvalues of J^* are $\lambda_1 = 0$ and $\lambda_{2,3} = (1 + \alpha) \pm i\beta$. It follows that the convergence condition is $(1 + \alpha)^2 + \beta^2 < 1$, where $\alpha = (\tau^2 \frac{k_0^2}{2} - \tau D + \tau^2 \frac{D^2}{2})$ and $\beta = (\tau k_0 - \tau^2 D)$, and this condition gives the inequality of the theorem. In particular, if $D = 0$ we have the inequality $k_0^4 < 0$ which is never satisfied. \square

Remark 3.5 *The convergence of the iteration scheme generated by the matrix J^* does not depend on $\tilde{\tau}$, θ_1 and θ_2 because these parameters do not appear in the eigenvalues of J^* and in the inequality of the theorem 3.3.*

The equation associated to the inequality of the theorem 3.3 is given by

$$a\tau^3 - b\tau^2 + c\tau - d = 0, \quad (32)$$

where $a = (k_0^4 + D^2(4 - k_0^2) + 4D^2)$, $b = 4D(D^2 + 2k_0 - k_0^2)$, $c = 8D^2$ and $d = 8D$. This equation has two complex conjugate solutions and a real solution given by

$$\tau_c = \frac{b}{3a} - \frac{2^{1/3}(-b^2 + 3ac)}{3aq^{1/3}} + \frac{1}{2^{1/3}3a}q^{1/3}, \quad (33)$$

where $q = 2b^3 - 9abc + 27a^2d + \sqrt{4(-b^2 + 3ac)^3 + (2b^3 - 9abc + 27a^2d)^2}$. It follows that the iteration scheme induced by the matrix J^* is convergent if and only if the following inequality is satisfied

$$\tau < \tau_c. \quad (34)$$

In particular, with the positions $g = 9.81 \text{ m/sec}^2$, $k_0 = 10^{-4} \text{ sec}^{-1}$, $k_1 = 40 \text{ m}^{1/2}\text{sec}^{-1}$, $|u| = 0.1 \text{ m/sec}$ and $H = 0.1 \text{ m}$, we find $\tau < \tau_c = 5.41 \text{ sec}$.

We shall use this estimation for the numerical implementation of the splitting algorithm.

In conclusion, we see that the convergence of the first part of the splitting method depends strongly on the time step τ . This time step must be very small to ensure the convergence of the iteration scheme. Moreover the inclusion of the Chezy force is essential for the convergence. In fact, if the Chezy force is not included ($D = 0$) then the iterative process diverges.

We observe that the problem of spatial discretization can be studied with the Finite Element Method (see next section). As shown by Ciarlet and Raviart¹¹, the interpolating function of Finite Element Method controls completely the spatial convergence. It follows that, to analyze the time convergence of the second part of the splitting method, the spatial dependence can be neglected. In this way, we get $(\Delta u_i^{**})^{(n+1)} = 0$, $i = 1, 2$, and the splitting scheme reads

$$\begin{aligned} \eta^{(n+1)} &= \eta^{(n)} + (\Delta \eta^{**})^{(n+1)} \\ u_1^{(n+1)} &= u_1^{(n)} + (\Delta u_1^*)^{(n+1)} \\ u_2^{(n+1)} &= u_2^{(n)} + (\Delta u_2^*)^{(n+1)} \end{aligned} \quad (35)$$

where

$$(\Delta \eta^{**})^{(n+1)} = -\tilde{\tau} \left[\frac{\partial H}{\partial x_1} + \frac{\partial H}{\partial x_1} \theta_1 \alpha - \frac{\partial H}{\partial x_2} \theta_1 \beta \right] u_1^{(n)} - \tilde{\tau} \left[\frac{\partial H}{\partial x_1} \theta_1 \beta + \frac{\partial H}{\partial x_2} + \frac{\partial H}{\partial x_2} \theta_1 \alpha \right] u_2^{(n)}$$

$$\begin{aligned}(\Delta u_1^*)^{(n+1)} &= \alpha u_1^{(n)} + \beta u_2^{(n)} \\(\Delta u_2^*)^{(n+1)} &= -\beta u_1^{(n)} + \alpha u_2^{(n)}\end{aligned}\tag{36}$$

with $\alpha = (-\frac{\tau^2 k_0^2}{2} - \tau D + \frac{\tau^2 D^2}{2})$, $\beta = (\tau k_0 - \tau^2 D)$ and $D = g|u|/(k_1^2 H)$. It follows that the iteration scheme can be written as

$$\begin{pmatrix} \eta^{(n+1)} \\ u_1^{(n+1)} \\ u_2^{(n+1)} \end{pmatrix} = J^{**} \begin{pmatrix} \eta^{(n)} \\ u_1^{(n)} \\ u_2^{(n)} \end{pmatrix}\tag{37}$$

where

$$J^{**} = \begin{pmatrix} 1 & -\tilde{\tau}[\frac{\partial H}{\partial x_1} + \frac{\partial H}{\partial x_2}\theta_1\alpha - \frac{\partial H}{\partial x_2}\theta_1\beta] & -\tilde{\tau}[\frac{\partial H}{\partial x_2} + \frac{\partial H}{\partial x_1}\theta_1\beta - \frac{\partial H}{\partial x_2}\theta_1\alpha] \\ 0 & 1 + \alpha & \beta \\ 0 & -\beta & 1 + \alpha \end{pmatrix}.\tag{38}$$

Remark 3.6 *The iteration scheme induced by the matrix J^{**} does not depend on θ_2 .*

It is easy to see that the eigenvalues of J^{**} are the same of J^* . They are $\lambda_1 = 0$ and $\lambda_{2,3} = (1 + \alpha) \pm i\beta$. It means that also the convergence of the iteration scheme generated by the matrix J^{**} does not depend on $\tilde{\tau}$, θ_1 and θ_2 because these parameters do not appear in the eigenvalues of J^{**} . In conclusion, we do not have limitations for $\tilde{\tau}$. In practice, because of the effect of the error propagation due to the numerical approximations, we use $\tilde{\tau} \simeq 5$ minutes.

4 Finite element space discretization

The spatial approximation with the finite element method is obtained by using linear form functions for the integer step $(n, n+1, \dots)$ and constant functions for the half step $(n-1/2, n+1/2, \dots)$. In this way the equation (13) reads (see also Ref. 12 and 13)

$$\begin{aligned}(\mathbf{M}\Delta \mathbf{U}^*)_j^{(n+1)} &= \tau \int_{\Omega} (\mathbf{R}_s)^{(n)} \phi_j d\mathbf{x} + \frac{\tau^2}{2} \int_{\Omega} \frac{2}{\tau} [(\mathbf{R}_s)^{(n+1/2)} - (\overline{\mathbf{R}}_s)^{(n)}] \phi_j d\mathbf{x} \\ &= \tau \int_{\Omega} \{ (\mathbf{R}_s)^{(n+1/2)} + [(\mathbf{R}_s)^{(n)} - (\overline{\mathbf{R}}_s)^{(n)}] \} \phi_j d\mathbf{x} \quad j = 1, 2, \dots, N\end{aligned}\tag{39}$$

where N is the total number of nodes, ϕ_j is the weight linear function of the node j , the bar denotes mean values calculated on the element and

$$\mathbf{M} = [M]_{ij} = \int_{\Omega} \phi_i \phi_j d\mathbf{x} \quad i, j = 1, 2, \dots, N \quad (40)$$

is the mass matrix.

By using again linear triangular elements for the spatial discretization with the finite elements method, and by using the Green formulas to the terms which include the second derivatives, the variational formulation (24) gives the following system of discrete equations

$$(\mathbf{M} + \tilde{\tau}^2 g \theta_1 \theta_2 \mathbf{S}) (\Delta p^{**})^{(n+1)} = -\tilde{\tau} \left\{ \sum_{i=1}^2 \mathbf{Q}_i \left[H \left(u_i^{(n)} + \theta_1 (\Delta u_i^*)^{(n+1)} \right) \right] + \tilde{\tau} \theta_1 \mathbf{S} p^{(n)} \right\} \quad (41)$$

where

$$\mathbf{S} = \sum_{i=1}^2 \left(\int_{\Omega} \frac{\partial[\phi]}{\partial x_i} H \frac{\partial[\phi]^T}{\partial x_i} d\mathbf{x} \right) \quad \text{and} \quad \mathbf{Q}_i = \int_{\Omega} [\phi] \frac{\partial[\phi]^T}{\partial x_i} d\mathbf{x} . \quad (42)$$

When $(\Delta \eta^{**})^{(n+1)}$ is evaluated by (41), the second and third equations of (23) can be solved in $(\Delta u_1^{**})^{(n+1)}$ and $(\Delta u_2^{**})^{(n+1)}$; their discretization to finite elements is given by

$$\mathbf{M}(\Delta u_i^{**})^{(n+1)} = -\tilde{\tau} g \mathbf{Q}_i \left[\eta^{(n)} + \theta_2 (\Delta \eta^{**})^{(n+1)} \right] \quad i = 1, 2 . \quad (43)$$

It is easy to see that this set of equations is similar to (39); as a consequence it can be solved in the same way.

In summary, the solution at each time step implies the following operations:

- a) solve (39) for $(\Delta u^*)^{(n+1)}$ and $(\Delta u^*)^{(n+1)}$ (here $(\Delta \eta^*)^{(n+1)} = 0$ because the first component of \mathbf{R}_s is zero);
- b) calculate $(\Delta \eta^{**})^{(n+1)}$ from (41);
- c) solve (43) for $(\Delta u_1^{**})^{(n+1)}$ and $(\Delta u_2^{**})^{(n+1)}$;
- d) calculate $\mathbf{U}^{(n+1)} = \mathbf{U}^{(n)} + (\Delta \mathbf{U}^*)^{(n+1)} + (\Delta \mathbf{U}^{**})^{(n+1)}$.

5 Numerical calculations

In this section we describe our calculations of the motion of the water into the Venice lagoon^{12,13}. The equations considered are the shallow water equations. All simplifications are obtained considering different depths of the lagoon and in particular a finer discretization is considered where the water runs with greater speed namely along the more important canals. The numerical method is based on the splitting scheme and the finite element discretization presented in the previous sections.

To analyze the Venice Lagoon we must add the wind effect. Let v the wind velocity respect the water and ξ an adimensional constant, the wind term is given by $\xi|v|v/H$. In the approximation that the water velocity is negligible respect the wind velocity, the wind term is an external parameter to the system, i.e. the wind velocity respect the soil. We take a time-dependent wind velocity but space-independent¹⁴.

Following our analytical estimation of the time step which is necessary for the convergence of the splitting scheme, we choose a time step $\tau = 3$ seconds for the explicit part and $\tilde{\tau} = 5$ minutes for the implicit part. In this way the model is very stable and capable to preserve the water quantity of the system. We decompose the Venice lagoon into 10 subdomains and the nodes of the interfaces collected into a unique set. The number of nodes of the entire discretization is 1967 and the elements are 3423. The results of the calculation obtained using the domain decomposition method without parallelization have approximately a speed-up of 25 per cent over a Conjugate Gradient solver used on a not-decomposed domain^{12,13,17}.

The model has been verified by using the experimental data of the Ufficio Idrografico and Mareografico di Venezia (month of February 1994)¹⁵. The data have been introduced at the mouth of the lagoon as boundary conditions on the open boundary. Therefore a comparison has been made between the simulation and the real data on the internal measurement stations of the lagoon¹⁶. The experimental and simulated data present a good agreement in all the internal measurement stations^{12,13,17}.

References

1. A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics* (Springer, Berlin, 1980).
2. L. Quartapelle, *Numerical Solution of the Incompressible Navier–Stokes Equations* (Birkhouse, Bessel, 1993).
3. R. Temam, *Navier-Stokes Equations* (North Holland, Amsterdam, 1977).
4. G.F. Carey and J.T. Oden, *Finite Elements: Fluid Mechanics* (Prentice–Hall, New Jersey, 1986).
5. J. Peraire, O.C. Zienkiewicz and K. Morgan, "Shallow water problems: a general explicit formulation", *Int. J. Numer. Meth. Eng.* **22**, 547–574 (1986).
6. M. Morandi Cecchi, "Study of the behaviour of oscillatory waves in a lagoon", *Int. J. Numer. Meth. Eng.* **27**, 103–112 (1989).
7. O.C. Zienkiewicz and P. Ortiz, "A split–characteristic based finite element model for shallow water equations", *Int. J. Numer. Methods Fluids* **20**, 1061–1080 (1995).
8. O.C. Zienkiewicz and P. Ortiz, "An Improved Finite Element Model for Shallow Water Problems", in *Finite Element Modelling of Enviromental Problems*, ch. 4, pp. 61–84 (Wiley, New York, 1995).
9. R. Glowinski, "Finite element methods for the numerical simulation of incompressible viscous flow. Introduction to the control of Navier–Stokes equations", *Lect. Appl. Math.* **28**, 219–301 (1991).
10. R. Temam, "Multilevel Methods for the Simulation of Turbulence", *J. Comput. Phys.* **127**, 309–315 (1996).
11. P.G. Ciarlet and P.A. Raviart, "General Lagrange and Hermite interpolation in R^n with applications to finite element methods", *Arch. Rational Mech. Anal.* **46**, 177–199 (1972).
12. M. Morandi Cecchi, A. Pica and E. Secco, "Tidal flow analysis with core minimization", in the 8th International Conference on Finite Elements in Fluids: Ed. by K. Morgan *et al.*, pp. 990–998 (Pineridge Press, U.K., 1993).
13. M. Morandi Cecchi, A. Pica and E. Secco, "Domain decomposition of a finite element model of a lagoon", in the 8th International Conference on Finite Elements in Fluids: Ed. by K. Morgan *et al.*, pp. 1016–1036 (Pineridge Press, U.K., 1993).
14. G. Di Silvio, "Modelli matematici per lo studio della propagazione di onde lunghe e del trasporto di materia nei corsi d'acqua e nelle zone costiere", in

Equazioni Differenziali dell'Idrologia e dell'Idraulica, Ed. by G. Casadei *et al.*, pp. 11–41 (Patron, Bologna, 1979).

15. L. D'Alpaos and G. Di Silvio, *Le correnti di marea nella Laguna di Venezia*, (Ministero dei Lavori Pubblici, 1979).

16. C.N.R. Grandi Masse, *Foglio accompagnatorio dei rilevamenti digitalizzati delle profondità nella laguna di Venezia* (C.N.R. Grandi Masse, Venezia, 1970).

17. M. Morandi Cecchi and L. Salasnich, "Shallow water theory and its application to the Venice lagoon", Preprint 3/97, Dep. of Pure and Appl. Math., Univ. of Padova, to be published in the Proceedings of the Symposium on Advances in Computational Mechanics, 13–15 January 1997, Austin (Texas).